

Improving Markov Chain Models for Road Profiles Simulation via Definition of States

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Abstract— Road profiles are a major excitation to the chassis and the resulting loads drive vehicle designs. The physical resources needed to measure, record, analyze, and characterize an entire set of real, spectrally broad roads is often infeasible for simulation. This motivates the need for more accurate models for characterizing roads and for generating synthetic road profiles of a specific type. First order Markov Chain models using uniform sized bins to define the states have been previously proposed to characterize and synthetically generate road profiles. This method, however, was found to be unreliable when the number of states is increased to improve resolution. In an effort to solve this problem, this work develops a method by which states are defined using non-uniform sized, percentile-based bins which results in a more fully populated transition matrix. A statistical test is developed to quantify the confidence with which the estimated transition matrix represents the true underlying stochastic process. The order of the Markov Chain representation of the original and synthetic profiles is checked using a series of preexisting likelihood ratio criteria. This method is demonstrated on data obtained at the Virginia Tech VTTI location and shows a considerable improvement in the estimation of the transition properties of the stochastic process. This is evidenced in the subsequent generation of synthetic profiles.

I. INTRODUCTION

CHASSIS development is an iterative process that requires that the designer have prior knowledge about input excitations in order to predict target chassis loads [1]. The main and continuous excitation to the chassis is the terrain or road over which a vehicle travels. By providing the designer with a set of consistent characteristic terrain profiles, more accurate predictions about chassis loads can be made. A consistent set of terrain profiles would allow the engineer to make informed decisions about the chassis design and simulate how those design options affect the peak

load estimation as well as long-term durability load analysis.

Measuring, recording, and analyzing large continuous sets of multiple types of roads for simulation is costly and infeasible. In this work, a measured road profile is considered a single realization of an underlying stochastic process. A measured road profile, therefore, is only one of many possible profiles that the vehicle could encounter. A mathematical framework must be developed to capture the properties of the underlying stochastic process, thereby characterizing the roads (realizations) that could be synthesized.

Many different models for road profiles have been tested and applied. The standard method used to characterize and classify road profiles is the power spectral density (PSD) [2], [3]. Some researchers have observed that the shape of the PSD is independent of the road type [4], whereas other researchers have shown, through individual example, a relationship between the dropping of the PSD amplitude for a large section of frequency bandwidth due to the removal of large spiked events in their data before PSD calculation [5]. The view of the present authors is more in line with the latter's observation, [5], that is the PSD is a generalized and global characterizing calculation, lumping both transient and long trend spatial analysis into one. Although PSD's and their approximations [3] have been standardized [6], the aim of this work is to apply a non-generalized first order Markov Chain model to a specific road profile, capture its specific properties, and employ the model to synthesize data with the same statistical properties.

Other techniques have been proposed to characterize specific properties of specific roads. The wavelet decomposition [7], and the Hilbert-Huang transform (HHT) [8] have both shown great promise in identifying and extracting specific transient events in road profiles [8] [9]. All of the previous methods, contribute to the observation that terrain and road profiles are a complex and diverse set of signals. This observation and the application of many different models lead to the suggestion of rigorously testing of statistical properties of a profile before the application of a statistical model [10]. Similarly, in this paper, the authors propose the verification of the first order Markov Chain properties before application of the model.

Markov Chains have been proposed as candidates for road profile models [1],[11]. However these methods were limited in their applicability when the transition matrix was sparsely populated. Currently, the state is defined by

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uniform sized bins (each bin captures an equal range of height variation in the terrain). Each transition probability can be estimated from a measured profile by calculating the ratio of the number of transitions from state i to state j to the number of departures from state j . Now consider another road profile. If the new profile is a realization of the same stochastic process, then the probability distribution for the number of transitions from state i to state j is defined by a binomial distribution. In this way, an increased number of data points for a given transition will result in a more confident estimate of the transition probability. In order to properly populate the estimated transition matrix, the states should be defined such that a significant number of data points occur within a state. This motivates a change in methods to define the states. Presently, this work develops a non-uniform, percentile-based state binning method to populate the estimated transition matrix. This results in a more representative approximation of the underlying transition matrix when it is being estimated from real data.

The method is demonstrated on a set of 89 measured two-dimensional road profiles, obtained at the Virginia Tech VTTI location. The data are used to generate first order Markov models and synthesize data. The resulting distributions of the transition probabilities for both uniform and non-uniform sized state definitions are compared. Additionally, the resulting models are tested for 0th and 1st order Markov Chain properties using statistical likelihood ratio criteria. Next, the appropriateness of these methods is tested using a statistical two sample Kolmogorov-Smirnov test of the binomial percentiles of data. Two conclusions can be drawn from the results of these statistical tests: 1) The confidence in the estimated transition matrix is much higher when using the non-uniform, percentile-based method developed in this work, and 2) The ability to represent the data accurately as a first order Markov Chain is also better when using the proposed method.

II. BACKGROUND

In this section, the relationships between Markov Chains, transition matrices, and binomial distributions are presented. Let a family of random variables X be a discrete time stochastic process such that $\{X_m, m = 0, 1, 2, \dots\}$. Let S denote the state space where state values $x_m \in S, \forall m$. The first order Markov property can be stated as:

$$\begin{aligned} P(X_{m+1} = x_{m+1} | X_m = x_m, X_{m-1} = x_{m-1}, \dots, X_0 = x_0) \\ = P(X_{m+1} = x_{m+1} | X_m = x_m) \end{aligned} \quad (1)$$

Equation (1) states that the future state x_{m+1} is conditioned only on the present state x_m . When the family of random variables X_m satisfy the first order Markov property and are shown to have stationary transition probabilities, then a first order Markov Chain for r states can (Unclassified: Dist A. Approved for public release)

be described by a single transition matrix [12].

$$\begin{aligned} P \triangleq p_{ij} &= p(X_{m+1} = j | X_m = i) \\ \forall i &= 1, \dots, r \text{ and } j = 1, \dots, r \in S \end{aligned} \quad (2)$$

The estimated transition matrix is calculated in the following manner. Let n_{ij} be the number of occurrences of transitions from state i to state j and let N be the matrix of all possible n_{ij} . Then the maximum likelihood estimate (MLE) of the true transition matrix [13] has been shown to be

$$\hat{P} \triangleq \hat{p}_{ij} = \frac{n_{ij}}{N_i} \quad \forall i, j \in S \quad (3)$$

where

$$N_i = \sum_j n_{ij}, \quad \forall i, j \in S \quad (4)$$

Now consider a new realization. If the realization comes from the underlying stochastic process with these estimated transition probabilities, then the probability distribution for the number of transitions from state i to state j , n_{ij} , is a discrete binomial distribution with probability mass function (PMF) [14].

$$\begin{aligned} n_{ij} &\sim b(n_{ij}; N_i, p_{ij}) \\ &= \binom{N_i}{n_{ij}} p_{ij}^{n_{ij}} (1 - p_{ij})^{N_i - n_{ij}} \end{aligned} \quad (5)$$

where p_{ij} is estimated by \hat{p}_{ij} . Note that the i^{th} row in the estimated transition matrix can be equivalently expressed as a multinomial distribution. Once the MLE (3) is obtained, synthetic profiles can be generated by uniformly randomly sampling from the binomial distribution. Let n'_{ij} be the number of transitions from state i to state j in the synthetic profile, where the prime denotes ‘‘synthetic’’. Then the synthetic profile is governed by

$$n'_{ij} \sim b(n'_{ij}; N'_i, \hat{p}_{ij}) \quad (6)$$

Fig. 1 illustrates that binomial distribution PMF’s and cumulative distribution functions (CDF’s) have a wide range of shapes which are dependent on the total number of trials and probability of an event occurring. Specifically, if there are an insufficient number of occurrences of a state N_i , then the estimate of the transition probability is poor and the sensitivity of the probability estimate is highly sensitive to small incremental changes in the number of occurrences. Consider the following two cases. In Case 1 the number of departures is five and the transition probability is 0.2. The expected number of occurrences is one and the change in the probability when the number of occurrences increases from one to two is 20%. In Case 2 the number of departures is 50 and the transition probability is 0.2. The expected number of occurrences is ten and the change in the probability when the number of transitions increases

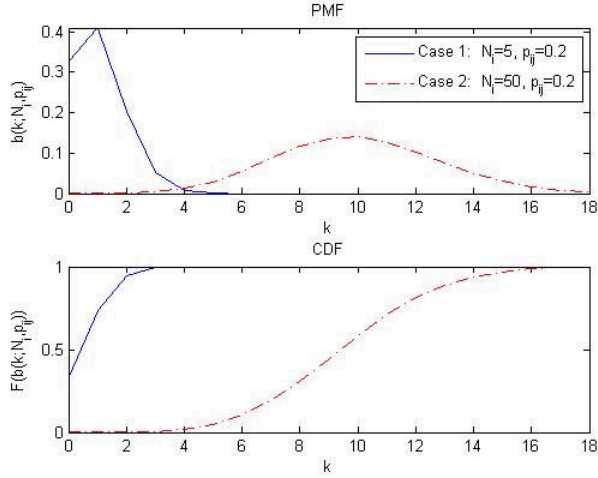


Fig. 1. Two case example: Binomial PMF's and CDF's showing sensitivity to distribution parameters

from ten to eleven is 1%. This example, graphically represented in Fig. 1, illustrates the effect of the number of occurrences on the confidence with which the transition probabilities can be calculated.

Let $F(b(n_{ij}; N_i, p_{ij}))$ be the CDF of the binomial distribution. Then the binomial percentiles are defined by

$$B \triangleq 100 \times F(b(n_{ij}; N_i, p_{ij})) \quad \forall i, j \in S \quad (7)$$

The differences between unpopulated and populated binomial distributions can also be described in terms of CDF percentiles. A percentile is a value of a variable that marks where a certain percent of the observations fall under. For example, the 40th percentile value indicates that 40% of the observations lie below the value. Case 1 is only uniquely defined for the n_{ij} interval from zero (33%) to six (100%) occurrences, whereas Case 2 is uniquely defined for the entire n_{ij} range of zero (0%) to eighteen (~100%). Specifically, if a state is not well populated, then the binomial distribution underlying the transition matrix is not well defined. This results in the limitation of the range of unique percentile values available to sample from when synthesizing data.

III. STATISTICAL TESTS

A. Likelihood Ratio Criterion Tests for Markov Chain Order

The application of a Markov Chain model requires that the Markov Chain order be verified. The problem of estimating the true transition matrix from data can be expressed using the likelihood principle. The likelihood principle quantifies how “likely” an unknown parameter is, conditioned on the given data. In this application, the unknown parameters are the individual entries of the true transition matrix. Applying the likelihood principle to this application, the likelihood (Unclassified: Dist A. Approved for public release)

function describes how likely the estimates of the transition matrix are, under a given model, and given the road profile data. Previously, [13] developed a likelihood ratio criterion for the Markov property and first order property [15]. The likelihood ratio is a statistic that compares the fit of two data models. The two Markov Chain order tests that were applied are summarized as follows. Test 1: The first likelihood ratio criterion tests the null hypothesis, H_\emptyset that the estimated transition matrix \hat{P} is 0th order versus the alternative hypothesis H_{Alt} that \hat{P} has first order properties using

$$\lambda = \prod_{i,j} [\hat{p}_j / \hat{p}_{ij}]^{n_{ij}} \quad (8)$$

The marginal probability is defined by $\hat{p}_j = \sum_i \hat{p}_{ij}$ and the required test statistic $-2 \ln \lambda$ is asymptotically χ^2 distributed with $(r-1)^2$ degrees of freedom under the null hypothesis [13],[15]. Test 2: The second likelihood ratio criterion tests the null hypothesis, H_\emptyset that \hat{P} is 1st order versus the alternative hypothesis, H_{Alt} that \hat{P} has second order properties using $k = 1, \dots, r$ and

$$\lambda = \prod_{i,j,k} [\hat{p}_{jk} / \hat{p}_{ijk}]^{n_{ijk}} \quad (9)$$

where

$$\begin{aligned} \hat{p}_{jk} &= \sum_i n_{ijk} / \sum_i \sum_k n_{ijk} \\ \hat{p}_{ijk} &= n_{ijk} / \sum_k n_{ijk} \end{aligned} \quad (10)$$

For this test, the required test statistic $-2 \ln \lambda$ is asymptotically χ^2 distributed with $r(r-1)^2$ degrees of freedom under the null hypothesis [13], [15]. This paper requires Test 1 to result in H_{Alt} and Test 2 to result in H_\emptyset simultaneously to confirm the first order Markov property. These likelihood ratio criterion tests are applied to the differenced measured data and the synthetic data for both uniform and non-uniform sized bin techniques.

B. Two Sample Kolmogorov-Smirnov Test for Estimated Transition Matrices

The discussion concerning the binomial percentiles and their relationship to the first order Markov transition matrix motivates the need for a statistical test to quantify how well synthetic realizations represent the estimated transition matrix. Consider the number of transitions in a synthetic profile from state i to state j . The number of occurrences is binomially distributed as developed in (5). The corresponding CDF for this distribution is given in (7). If the synthetic data is generated from the transition matrix, then one would expect that 50% of the transitions in the

matrix would have a CDF of less than 50%. Similarly, one would expect that $P\%$ of the transitions would have a CDF less than $P\%$. Simply put, if the synthesized profile is a realization of the transition matrix, then the distribution of these CDFs should be uniformly distributed themselves. Consider a specific example where seven states are used, creating 49 transitions. For a given synthetic profile the number of transitions follows a binomial distribution and has a corresponding CDF (5) and (7). Now consider sorting all 49 of these CDFs. If the synthetic profiles come from the transition matrix, then the CDFs should be (nearly) uniformly distributed between 0% and 100%.

The acceptable amount of deviation from this straight line can be testing using a two sample Kolmogorov-Smirnov (K-S) test that compares the empirical CDF of the actual binomial percentiles sampled to the reference CDF (a straight line). The null hypothesis is that the synthetic profile comes from the transition matrix. Again, to be clear, the distribution of the CDFs of the number of occurrences for *all* transitions should be uniform. Underlying that, the number of occurrences for *each* transition is binomially distributed.

The K-S test is applied as follows. Let $B'_{U \text{ or } NU}$ be the binomial percentiles (7) of the synthetic data (uniform or non-uniform binning method), where the prime denotes synthetic hereafter.

$$B'_{U \text{ or } NU} \triangleq 100 \times F\left(b\left(n'_{ij}; N'_{ij}, \hat{p}_{ij}\right)\right) \quad (11)$$

$$\forall i, j \in S$$

Let B_{ref} be the reference uniform distribution of binomial percentiles, which occur at r^2 evenly spaced points on the interval $[0,100]$. The max distance between the empirical and reference CDF's is

$$D = \max_{[0,100]} |F(B'_{U \text{ or } NU}) - F(B_{ref})| \quad (12)$$

Then the null hypothesis H_0 is rejected at a significance level α if

$$\sqrt{r/2} D > C(\alpha) \quad (13)$$

where $C(\alpha)$ is the critical value at the specified value of α .

IV. EXAMPLE

In this section, an example comparing uniform and non-uniform binning state defining techniques is presented. First the measured data is checked to see if a first order Markov Chain model is applicable using the autocorrelation function. In order for a first order Markov model to apply, the autocorrelation [16] must not be significant past the first lag [11],[17]. Fig. 2 a) illustrates the original and differenced data and Fig. 2 b) indicates that the differenced road profile is a good candidate for a first order Markov Chain model.

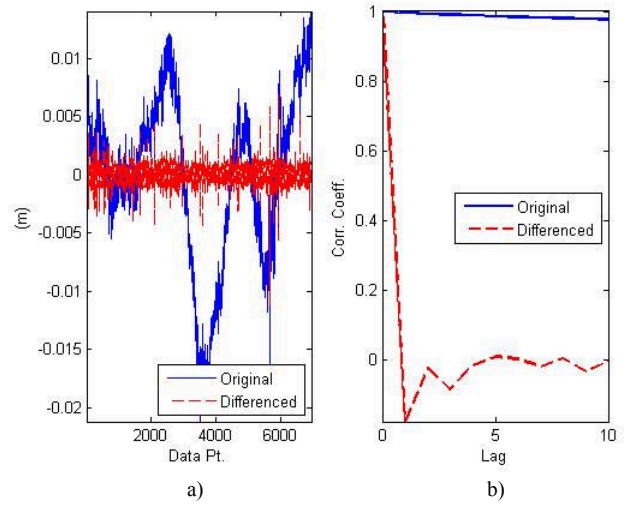


Fig. 2. a)Original and differenced data, b)Autocorrelation of original and differenced data

Next the differenced data was binned into $r = 7$ states using the current method of uniform binning and the proposed method of non-uniform percentile-based sized bins. The contrast between the two binning methods is illustrated by considering the CDF of the differenced data. Fig. 3 shows that the uniform binning method produces equally sized bins in terms of the range of the data on the x-axis (shown as solid vertical lines), while the proposed non-uniform percentile-based method produces equally sized bins in terms of the cumulative probability on the y-axis (shown as dash-dot horizontal lines indicating equal probability intervals and corresponding vertical dash-dot lines showing how the equal probability intervals correspond to intervals in the state space). The binned data were then tested using the likelihood ratio criterion tests for Markov Chain order and were found to have the first order Markov Chain property.

Next, the number of occurrences of all state transitions, N , and estimated transition matrices \hat{P} were calculated. Fig. 4 a)-b) illustrate that the uniform binning of states does not fully populate N and \hat{P} . This results in the loss of the ability to synthesize profiles accurately from \hat{P} due to the not well defined binomial distributions as discussed in the background section. Fig. 4 c)-d), however, illustrate that the non-uniform percentile-based binning of states fully

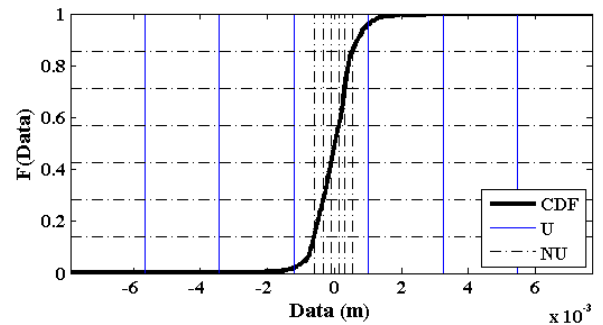


Fig. 3. Bin edges using uniform (U) and non-uniform (NU) percentile-based methods and CDF of differenced data

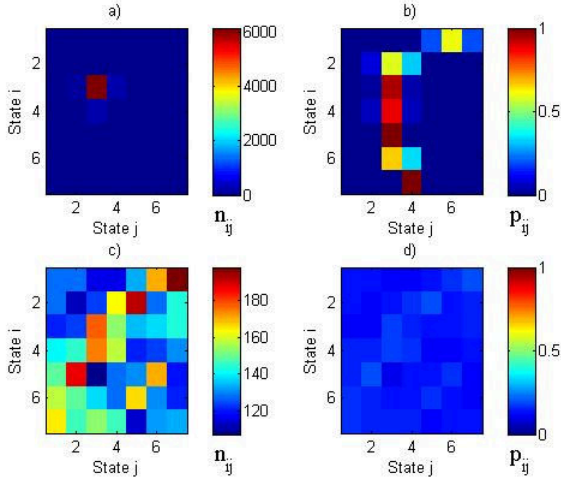


Fig. 4. a)Uniform bins N , b)Uniform bins \hat{P} , c)Non-Uniform percentile-based bins N , d) Non-Uniform percentile-based bins \hat{P} populates N and \hat{P} . The non-uniform percentile based method results in well defined underlying binomial distributions which enables accurate sampling for synthesizing profiles.

Furthermore, the numbers in N using both binning methods can be summarized below by Fig. 5. Fig. 5 is a histogram of *all* n_{ij} for both methods. Specifically, Fig. 5 shows that the entire matrix of state transitions N is governed by the binomial distribution. Fig. 5 illustrates that the non-uniform binning method has a much wider range of n_{ij} occurrences. Recalling the example illustrated in Fig. 1 and (5), it can be concluded that the uniform binning method is similar to Case 1 (not well defined binomial distribution), and the non-uniform method is similar to Case 2 (well defined binomial distribution). This implies that synthesizing profiles using the uniform binning method will produce inaccurate samples due to the not well defined binomial distribution.

Next, the transition matrices are used to generate synthetic realizations. The realizations are quantized using the two

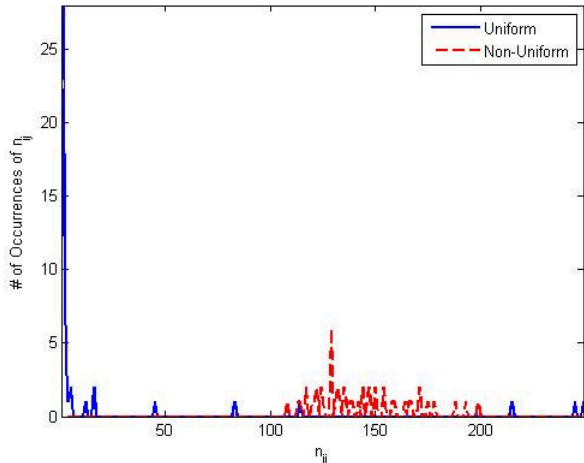


Fig. 5. Number of occurrences of n_{ij} vs. n_{ij} from counting matrices

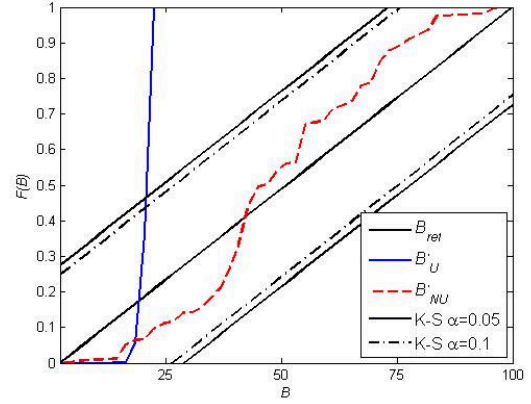


Fig. 6. K-S tests for CDFs: current method of uniform binning and proposed method of percentile-based binning compared to reference CDF ($r = 7$ states)

binning methods: the current uniform binning method and the proposed non-uniform percentile-based binning method. The corresponding number of transitions is determined as well as the binomial percentile CDF's (shown above in Fig. 6).

The two- sample K-S Test with significance level of $\alpha = 0.05$ is applied to the CDF of the number of occurrences using uniform binning and non-uniform binning. The results in Fig. 6 indicate that the proposed non-uniform binning method approximates uniform distribution of the CDFs and the K-S fails to reject the null hypothesis. Fig. 6 also shows that the current practice of uniform binning results in a highly biased distribution where a majority of CDFs are in the 100th percentile, which is not representative of the underlying CDFs. Using the uniform binning method, the null hypothesis can be rejected, indicating that this method did not sample properly due to the not well defined binomial distribution.

V. DISCUSSION

The uniform and non-uniform percentile-based binning methods were applied to 89 measured and differenced road profiles obtained at the Virginia Tech VTTI location. The longitudinal resolution of the profiles was 2.5 cm which resulted in a total profile length of 173.3 m. The first order Markov Chain properties were found to hold for measured differenced and synthesized differenced data using the current method of uniform sized bins and the proposed method of non-uniform percentile-based sized bins. Fig. 7, however, shows that the proposed method is more accurate for synthesizing profiles as is evidenced in the mean and peak amplitude values. This is a direct result of the non-uniform percentile-based method capturing the spatial characteristics of the differenced data more accurately than the current uniform sized binning method.

Although this paper has shown substantial improvements to first order Markov Chain models for synthesizing differenced profiles, more work needs to be done to model the original data. Here, the differenced data was used because the original data low frequencies were difficult to

model using a first order Markov Chain. Since differencing the data is a specific, non-ideal high pass filter, the work presented in this paper serves as a benchmark proof of concept for applying Markov Chain models to spectrally decomposed profile data. Future work will concentrate on band-passed filtered data, using a small number of separate first order Markov Chains to model the original profiles. The implicit assumption of this future work is that the different frequency ranges are independent. The present work, however, makes two key contributions: 1) a method to more fully populate the transition matrix, 2) a two sample statistical test to determine how well the estimated transition matrix was sampled from, using the definition of the underlying binomial distribution.

VI. CONCLUSIONS

A comparison of two different binning methods to define the states for a first order Markov Chain model for differenced synthetic road profile data is developed in this work. The theory is developed and demonstrated through an example to show that the non-uniform percentile-based state definition results in a more fully populated estimated transition matrix.

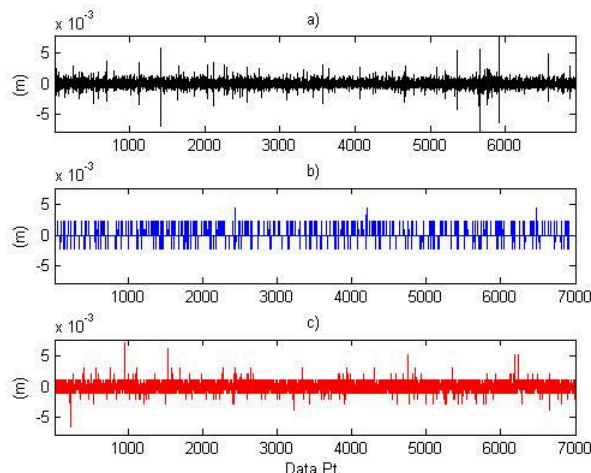


Fig. 7. Differenced data: a) measured, b) synthetic using uniform method, c) synthetic using non-uniform percentile-based method

This new method is also shown to result in more accurate sampling from the underlying binomial distribution which defines each entry in the transition matrix. A statistical test is developed to test how well a synthetic realization corresponds to the uniformly distributed CDF of number of transitions (where the number of transitions between a particular pair of states is binomially distributed). The results combine to indicate that use of the proposed non-uniform method results in more confident estimates of the transition matrix and more representative synthetic road profiles.

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